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Euler-Lagrange Equations with Generalized ABC and ABR Nabla Fractional Differences

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Abstract

Background: Discrete fractional calculus has emerged as an important mathematical framework for modeling dynamical systems with memory and nonlocal behavior. In particular, nabla fractional differences with Mittag–Leffler kernels have been widely used in discrete fractional variational problems and their associated Euler–Lagrange equations. **Objectives:** This study aims to derive new discrete fractional Euler–Lagrange equations involving generalized Atangana–Baleanu Caputo (ABC) and Atangana–Baleanu Riemann (ABR) nabla fractional differences with generalized Mittag–Leffler kernels. **Methods:** First, new summation-by-parts formulas for generalized ABC and ABR nabla fractional differences with a three-parameter Mittag–Leffler kernel are established. These formulas are then employed within the framework of discrete fractional calculus of variations to derive Euler–Lagrange equations for functionals containing generalized ABC nabla left and right fractional differences. **Results:** New discrete fractional Euler–Lagrange equations are obtained for variational problems involving generalized ABC nabla fractional differences with generalized Mittag–Leffler kernels. The proposed results extend previously known Euler–Lagrange formulations based on one-parameter kernels. Illustrative examples are presented to demonstrate the applicability of the theoretical findings. **Conclusions:** The results provide a generalized framework for discrete fractional variational problems and contribute to the development of Euler–Lagrange theory with generalized ABC and ABR nabla fractional operators, opening new directions for future research in discrete fractional calculus.

Keywords: Discrete fractional Euler–Lagrange equations; nabla fractional differences; discrete nabla Mittag–Leffler function; generalized ABC nabla fractional difference; summation by parts; discrete calculus of variations.

1. Introduction

The curiosity about continuous fractional calculus, in which the orders of differentiation and integration may be any real or complex numbers, is as old as ordinary calculus; it began in 1695 when Guillaume de l'Hôpital asked Gottfried Wilhelm Leibniz about the one-half derivative of x. The answer led to the name fractional calculus, which was considered a misnomer because the order of the derivative could be any number, not only rational numbers. Nevertheless, the firm foundation of continuous fractional calculus was not established until the late nineteenth century [1]. Due to its advantages and widespread applications in physics [2], medical science and computational biology [3], economics [4], control theory [5], and epidemiology [6], continuous fractional calculus has been studied extensively since the nineteenth century [7]. In addition, fractional-order derivatives have proven to be more suitable for modeling scientific and engineering problems than integer-order derivatives [7, 8]. This could be attributed to the fact that unlike integer-order derivatives and integrals, fractional operators are non-local, meaning that they depend on all historical values of the function, not only on its value at a single point. An encyclopedic reference for the different definitions and applications is provided in [8]. Nevertheless, the study of discrete fractional calculus has not attracted the same interest until quite recently.

Broadly speaking, discrete fractional calculus is defined as the study of sums and differences of arbitrary order, i.e., $\Delta^{\alpha} f(x)$ and $\nabla^{\alpha} f(x)$ on time scales—non-empty closed subsets of $\mathbb{R}$—where $\alpha \in \mathbb{R}$. There are two types of difference operators: the delta operator, $\Delta f(x) = f(x+1) - f(x)$, which is the forward difference operator, and the nabla operator, $\nabla f(x) = f(x) - f(x-1)$, which is the backward difference operator. In recent years, discrete fractional calculus has been applied successfully in various fields [9, 10], for instance in modeling porous materials [11], in signal processing and diffusion problems [12], and in cancer tumor growth [13].

The earliest definition of the discrete fractional difference, to the best of the authors' knowledge, appeared in 1910 by Chapman [14] as an infinite series known as the delta fractional sequential difference; however, serious interest in discrete fractional sums and differences defined on the time-scale sets $\mathbb{N}_a$ was stimulated by the pioneering work of Miller and Ross [1]. Other contributors to the theory of discrete delta fractional calculus include Atici and Eloe [15], and Abdeljawad and Baleanu [16].

This paper lies within the scope of discrete nabla fractional differences. Gray and Zhang in 1988 [17] were the first to introduce the definition of the nabla left fractional sum as a finite sum based on the repeated summation formula of $f(t)$, which is analogous to Cauchy's formula for repeated integrals; they also defined the corresponding fractional difference. Later, Abdeljawad and Atici in [18] observed that starting the summation from $a+1$ instead of a in the definition of the nabla left fractional sum proposed by Gray and Zhang is more consistent with the general framework of fractional calculus on time scales; hence, they refined the nabla left fractional sum operator defined in [17]. Furthermore, they introduced the nabla right fractional sum and difference. Correspondingly, Ahrendt in [19] established a unification of the nabla left fractional sum and difference introduced in [18]. All these definitions of fractional differences are called the Riemann–Liouville nabla fractional differences. There are other definitions of the nabla fractional difference, known as the Caputo nabla fractional differences. Abdeljawad in [20] further introduced new definitions of the Caputo nabla left and right fractional differences.

In continuous case, a new fractional difference was obtained by Caputo and Fabrizio in [21] by replacing the kernel of the integration with an exponential function, which is non-singular, to better describe some systems dynamics and the behavior of some materials. Subsequently, another definition was established by Atangana and Baleanu in [22] just by changing the exponential kernel with the M–L function since it is considered as a generalization of the exponential function. Following the same procedure in discrete fractional setting, Abdeljawad and Baleanu [23] introduced new definitions of the nabla fractional differences and sums, where the kernel of the summation in the fractional differences is non-singular with the nabla M-L function of one parameter. These new definitions are called the Atangana-Baleanu Caputo (ABC) nabla fractional differences, the Atangana-Baleanu Riemann (ABR) nabla fractional differences, and the Atangana-Baleanu (AB) nabla fractional sums. Subsequently, Abdeljawad and Baleanu in [24] traced the steps of continuous fractional calculus and introduced the nabla fractional differences and sums with the discrete nabla exponential kernel, which are called the Caputo–Fabrizio Caputo (CFC) nabla fractional differences and Caputo–Fabrizio Riemann (CFR) nabla fractional differences and the Caputo–Fabrizio (CF) nabla fractional sums. Later on, Abdeljawad [25] generalized the definitions given in [23] with the generalized nabla M-L function of three parameters.

Variational principles could be applied nicely in the framework of the discrete fractional calculus. Calculus of variations is a generalization of the theory of finding extrema of a function of several independent variables, where one seeks a vector in $\mathbb{R}^n$ such that the function $f: \mathbb{R}^n \to \mathbb{R}$ attains an extremum. In the calculus of variations, a functional is optimized rather than a function. Hence, instead of finding a point $(x_1, x_2, \dots, x_n)$ at which the function $f(x_1, x_2, \dots, x_n)$ has a maximum or minimum, one aims to find a function at which the functional attains an extremum [26]. The first discrete delta difference fractional Euler-Lagrange equation was derived by Atici and Sengul [13]. Then, Abdeljawad and Baleanu in [16] introduced different versions of the delta fractional Euler-Lagrange equation using a different definition of the delta right fractional difference. After that, Abdeljawad [27] established the first nabla Euler-lagrange equations. More recently, Abdeljawad and Baleanu [23] derived the ABC nabla fractional Euler-Lagrange equation. And naturally the Euler-Lagrange equations with Caputo-Fabrizio fractional differences were introduced by Abdeljawad and Baleanu in [24].

Since Abdeljawad in [25] introduced the generalized ABC nabla left fractional difference, the main objective of this paper is to obtain summation by parts identities for the generalized ABC nabla fractional differences and use them to derive the generalized ABC nabla fractional Euler-Lagrange equations.

2. Preliminaries

In this section, some basic definitions related to nabla calculus were given. Throughout this paper, the function $f(t)$ was considered real-valued and defined on the following sets:

$$\mathbb{N}_a =: \{a,\ a+1,\ a+2,\ \dots\},$$

where $a \in \mathbb{R}$, or

$$_b\mathbb{N} = \{b,\ b-1,\ b-2, \dots\},$$

where $b \in \mathbb{R}$, or

$$\mathbb{N}_a^b =: \{a,\ a+1,\ a+2,\ \dots,\ b\},$$

where $a, b \in \mathbb{R}$ and $b > a$.

The gamma function is an essential tool, which is used to define the discrete fractional operators and it is defined as follows:

Definition 2.1 (Gamma function). [28]

The gamma function can be expressed as follows: for any $z \in \mathbb{C}$, $\Re(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Definition 2.2 (Forward Jump Operator). [29] Let $x \in \mathbb{N}_a$. The forward jump operator σ is defined as

$$\sigma(x) := x + 1$$

Also, let $f^\sigma = f \circ \sigma$.

Definition 2.3 (Backward Jump Operator). [29] The backward jump operator ρ is defined on $\mathbb{N}_{a+1}$ as

$$\rho(x) := x - 1$$

Also, let $f^\rho = f \circ \rho$.

Definition 2.4 (Rising Factorial). [28] Let $n \in \mathbb{N}$. For any $t \in \mathbb{R}$, the rising factorial function $t^{\overline{n}}$ takes the form:

$$t^{\overline{n}} := t(t+1)(t+2) \dots (t+n-1)$$

Also, $t^{\overline{0}} := 1$.

Definition 2.5 (Generalized Rising Factorial). [28] The generalized rising factorial function is defined as follows:

$$t^{\overline{\alpha}} = \begin{cases} \frac{\Gamma(t+\alpha)}{\Gamma(t)}, & \text{if} t, t+\alpha \notin -\mathbb{N}_0, \\ 1, & \text{if} t = \alpha = 0, \\ 0, & \text{if} t \in -\mathbb{N}_0 \text{and} t+\alpha \in \mathbb{Z}^+, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Definition 2.6 (Delta Difference Operator). [1] Let $f: \mathbb{N}_a^b \to \mathbb{R}$. The delta operator Δ, also called the forward difference operator, is defined as follows:

$$\Delta f(x) = f(x+1) - f(x), \quad x \in \mathbb{N}_a^{b-1}.$$

Moreover, for $n \in \mathbb{Z}^+$,

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)), \quad x \in \mathbb{N}_a^{b-n}.$$

Also, $\Delta^0 f(x) = f(x)$; the identity operator.

Definition 2.7 (Nabla Difference Operator). [29] Let $f: \mathbb{N}_a \to \mathbb{R}$. The nabla operator, also called the backward difference operator ∇, is defined as:

$$\nabla f(x) = f(x) - f(x-1), \quad x \in \mathbb{N}_{a+1}.$$

Moreover, for $n \in \mathbb{Z}^+$,

$$\nabla^n f(x) = \nabla(\nabla^{n-1} f(x)), \quad x \in \mathbb{N}_{a+n}.$$

Also, $\nabla^0 f(x) = f(x)$; the identity operator.

The following two theorems state the delta and the nabla summation by parts formulas, which allow moving the delta or the nabla operator from one function to another:

Theorem 2.1 (Delta Summation by Parts). [28] Consider real-valued functions on $\mathbb{N}_a^b$, and let $c, d \in \mathbb{N}_a$ with $c < d$. The delta integration by parts formulas can be stated as follows:

$$\sum_{t=c}^{d-1} g(\sigma(t)) \Delta f(t) = g(t) f(t)|_c^d - \sum_{t=c}^{d-1} f(t) \Delta g(t). \quad (1)$$

Theorem 2.2 (Nabla Summation by Parts). [28] Let $f, g: N_a^b \to \mathbb{R}$ be two functions and $b, c \in N_a$ with $c < d$. The nabla integration by parts formulas are:

$$\sum_{t=c+1}^{d} g(\rho(t))\nabla f(t) = g(t)f(t)|_c^d - \sum_{t=c+1}^{d} f(t)\nabla g(t). \qquad (2)$$

Before moving to the definitions of the fractional differences, let us introduce the M-L function. In the continuous setting, the classical M-L function generalizes the exponential function, which is a solution of a linear ordinary differential equation with constant coefficients. Similarly, the M-L function serves as a solution of a fractional order differential equation [30]. In discrete fractional calculus, the discrete M-L function is a solution of a fractional difference equation [20].

Definition 2.8 (Discrete Nabla M-L). [20] Assume that $\alpha, \beta, t \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\forall \lambda \in \mathbb{R}$ with $|\lambda| < 1$. The discrete nabla M-L function of two parameters α and β takes the form:

$$E_{\overline{\alpha,\beta}}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{k\alpha+\beta-1}}}{\Gamma(k\alpha+\beta)}. \quad (3)$$

Notice that for $\beta = 1$, the M-L function of one parameter α is obtained:

$$E_{\overline{\alpha}}(\lambda, t) := E_{\overline{\alpha,1}}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{k\alpha}}}{\Gamma(k\alpha+1)}. \qquad (4)$$

Definition 2.9 (Nabla Discrete Generalized M–L). [20] Assume that $\alpha, \beta, \gamma, t \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\forall \lambda \in \mathbb{R}$ with $|\lambda| < 1$. The discrete nabla generalized M-L function of three parameters α, β, γ is expressed as:

$$E_{\overline{\alpha,\beta}}^{\gamma}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{k\alpha+\beta-1}}\,\gamma^{\overline{k}}}{\Gamma(k\alpha+\beta)k!}. \qquad (5)$$

Notice that if $\gamma = 1$, the discrete M-L function of two parameters for the nabla operator is recovered:

$$E_{\overline{\alpha,\beta}}^{1}(\lambda, t) = E_{\overline{\alpha,\beta}}(\lambda, t).$$

Theorem 2.3 [25] For $n = 1,2,...$, the following relations hold:

- $\nabla^n E_{\alpha,\beta+n}^{\rho}(\lambda, z) = E_{\alpha,\beta}^{\rho}(\lambda, z)$;
- $\nabla E_{\alpha}(\lambda, z) = \lambda E_{\alpha,\alpha}(\lambda, z)$.

The core of discrete fractional calculus lies in the definitions of the discrete fractional operators. These definitions are not unique, which gives the subject some flexibility. Some of these definitions are outlined as follows:

Definition 2.10 (Nabla Left and Right Fractional Sums)

1. [18] Let $f: \mathbb{N}_a \to \mathbb{R}, \alpha \in \mathbb{R}^+$. Then the nabla left fractional sum of order α can be defined as follows:

$$_a\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=a+1}^{t} (t-\rho(k))^{\overline{\alpha-1}} f(k), \ t \in \mathbb{N}_{a+1}, \qquad (6)$$

2. [18] Let $f:_b\mathbb{N} \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Then the nabla right fractional sum of order α can be defined as follows:

$$\nabla_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=t}^{b-1} (k-\rho(t))^{\overline{\alpha-1}} f(k), \ t \in_{b-1} \mathbb{N} \qquad (7)$$

Definition 2.11 (Riemann-Liouville Nabla Left and Right Fractional Differences)

1. [18] Let $f:\mathbb{N}_a\to\mathbb{R}, \alpha\in\mathbb{R}^+$ and let $N\in\mathbb{N}$ such that $N=[\alpha]+1$. Then the Riemann-Liouville nabla left fractional difference of order α can be defined as follows:

$$_a\nabla^{\alpha}f(t)=\nabla^{N}{}_a\nabla^{-(N-\alpha)}f(t),\ t\in\mathbb{N}_{a+1}, \qquad (8)$$

$$=\frac{\nabla^N}{\Gamma(N-\alpha)}\sum_{k=a+1}^{t}(t-\rho(k))^{\overline{N-\alpha-1}}f(k), \qquad (9)$$

The authors [19] stated the following alternative definition when $\alpha\notin\mathbb{N}$:

$$_a\nabla^{\alpha}f(t)=\frac{1}{\Gamma(-\alpha)}\sum_{k=a+1}^{t}(t-\rho(k))^{\overline{-\alpha-1}}f(k),\quad \alpha\in\mathbb{R}^+\backslash\mathbb{N},\ t\in\mathbb{N}_{a+1}, (10)$$

2. [18] Let $f:{}_b\mathbb{N}\to\mathbb{R}, \alpha\in\mathbb{R}^+$ and let $N\in\mathbb{N}$ such that $N=[\alpha]+1$. Then the Riemann-Liouville nabla right fractional difference of order α is defined as follows:

$$\nabla_b^{\alpha}f(t)=(-1)^N\Delta^N\nabla_b^{-(N-\alpha)}f(t),\ t\in{}_{b-1}\mathbb{N} \qquad (11)$$

$$=\frac{(-1)^N\Delta^N}{\Gamma(N-\alpha)}\sum_{k=t}^{b-1}(k-\rho(t))^{\overline{N-\alpha-1}}f(k), (12)$$

Moreover, when $\alpha\notin\mathbb{N}$

$$\nabla_b^{\alpha}f(t)=\frac{1}{\Gamma(-\alpha)}\sum_{k=t}^{b-1}(k-\rho(t))^{\overline{-\alpha-1}}f(k),\quad \alpha\in\mathbb{R}^+\backslash\mathbb{N},\ t\in{}_{b-1}\mathbb{N}. \qquad (13)$$

Now, the nabla fractional differences with discrete nabla M–L kernel of one parameter in the Riemann sense are expressed as follows:

Definition 2.12 (ABR Nabla Left and Right Fractional Differences). [23]

1. Let $f:\mathbb{N}_a\to\mathbb{R}, \alpha\in[0,1], \lambda=\frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the ABR nabla left fractional difference is defined as follows:

$$({}^{ABR}{}_a\nabla^{\alpha}f)(t)=\nabla_t\frac{B(\alpha)}{1-\alpha}\sum_{s=a+1}^{t}E_{\overline{\alpha}}(\lambda,t-\rho(s))f(s),\quad t\in\mathbb{N}_a. (14)$$

2. Let $f:{}_b\mathbb{N}\to\mathbb{R}, \alpha\in[0,1], \lambda=\frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the ABR nabla right fractional difference is defined as follows:

$$({}^{ABR}\nabla_b^{\alpha}f)(t)=\Delta_t\frac{-B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}E_{\overline{\alpha}}(\lambda,s-\rho(t))f(s),\quad t\in{}_b\mathbb{N}. \qquad (15)$$

Notice that the t in Δ_t and ∇_t indicates that the delta and nabla differences are with respect to t.

next, the nabla fractional differences with discrete nabla M–L kernel of one parameter in the Caputo sense are presented as follows:

Definition 2.13 (ABC Nabla Left and Right Fractional Differences). [23]

1. Let $f:\mathbb{N}_a\to\mathbb{R}, \alpha\in[0,1], \lambda=\frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the ABC nabla left fractional difference is defined as follows:

$$({}^{ABC}{}_a\nabla^{\alpha}f)(t)=\frac{B(\alpha)}{1-\alpha}\sum_{s=a+1}^{t}E_{\overline{\alpha}}(\lambda,t-\rho(s))\nabla_s f(s),\quad t\in\mathbb{N}_a. (16)$$

2. Let $f:_{b}\mathbb{N}\to\mathbb{R}, \alpha\in[0,1], \lambda=\frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the ABC nabla right fractional difference is defined as follows:

$$({}^{ABC}\nabla_{b}^{\alpha}f)(t)=\frac{-B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}E_{\overline{\alpha}}(\lambda,s-\rho(t))\Delta_{s}f(s),\quad t\in_{b}\mathbb{N}. \qquad (17)$$

Notice that the s in Δ_s and ∇_s indicates that the delta and nabla differences are with respect to s.

The following definitions are the generalized Atangana-Baleanu Riemann (ABR) nabla fractional differences, the generalized Atangana-Baleanu Caputo (ABC) nabla fractional differences, and the generalized Atangana-Baleanu (AB) nabla fractional sums. These definitions were obtained by changing the kernel of summation in the fractional difference operators with the M–L function of three parameters:

First, the generalized nabla fractional differences with discrete nabla M–L kernel of three parameters in the Riemann sense are defined as:

Definition 2.14 (Generalized ABR Nabla Left and Right Fractional Differences). [25]

1. Let $f:\mathbb{N}_a\to\mathbb{R}, \alpha\in(0,\frac{1}{2}), \lambda=\frac{-\alpha}{1-\alpha}$, $\gamma\in\mathbb{R}$, $\mathrm{Re}(\mu)>0$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the generalized ABR nabla left fractional difference is defined as follows:

$$({}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}f)(t)=\nabla_{t}\frac{B(\alpha)}{1-\alpha}\sum_{s=a+1}^{t}E_{\alpha,\mu}^{\gamma}(\lambda,t-\rho(s))f(s),\quad t\in\mathbb{N}_a. \qquad (18)$$

2. Let $f:_{b}\mathbb{N}\to\mathbb{R}, \alpha\in(0,\frac{1}{2}), \lambda=\frac{-\alpha}{1-\alpha}$, $\gamma\in\mathbb{R}$, $\mathrm{Re}(\mu)>0$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the generalized ABR nabla right fractional difference is defined as follows:

$$({}^{ABR}\nabla_{b}^{\alpha,\mu,\gamma}f)(t)=\Delta_{t}\frac{-B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}E_{\overline{\alpha,\mu}}^{\gamma}(\lambda,s-\rho(t))f(s),\quad t\in_{b}\mathbb{N}. \qquad (19)$$

Notice that the t in Δ_t and ∇_t indicates that the delta and nabla differences are with respect to t.

Next, the generalized nabla fractional differences with discrete nabla M–L kernel of three parameters in the Caputo sense can be presented as follows:

Definition 2.15 (Generalized ABC Nabla Left and Right Fractional Differences). [25]

1. Let $f:\mathbb{N}_a\to\mathbb{R}, \alpha\in(0,\frac{1}{2}), \lambda=\frac{-\alpha}{1-\alpha}$, $\gamma\in\mathbb{R}$, $\mathrm{Re}(\mu)>0$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the generalized ABC nabla left fractional difference is defined as follows:

$$({}^{ABC}{}_{a}\nabla^{\alpha,\mu,\gamma}f)(t)=\frac{B(\alpha)}{1-\alpha}\sum_{s=a+1}^{t}E_{\alpha,\mu}^{\gamma}(\lambda,t-\rho(s))\nabla_{s}f(s),\quad t\in\mathbb{N}_a. \qquad (20)$$

2. Let $f:_{b}\mathbb{N}\to\mathbb{R}, \alpha\in(0,\frac{1}{2}), \lambda=\frac{-\alpha}{1-\alpha}$, $\gamma\in\mathbb{R}$, $\mathrm{Re}(\mu)>0$, and let B be a normalization function such that $B(0)=B(1)=1$. Then the generalized ABC nabla right fractional difference is defined as follows:

$$({}^{ABC}\nabla_{b}^{\alpha,\mu,\gamma}f)(t)=\frac{-B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}E_{\overline{\alpha,\mu}}^{\gamma}(\lambda,s-\rho(t))\Delta_{s}f(s),\quad t\in_{b}\mathbb{N}. \qquad (21)$$

Notice that the s in Δ_s and ∇_s indicates that the delta and nabla differences are with respect to s.

Finally, the generalized nabla fractional sums with discrete nabla M–L kernel of three parameters are expressed as follows:

Definition 2.16 (Generalized AB Nabla Left and Right Fractional Sums). [25]

1. Let $f:\mathbb{N}_a \to \mathbb{R}, \alpha \in (0,\frac{1}{2}), \lambda = \frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0) = B(1) = 1$. Then the generalized AB nabla left fractional sum is defined as follows:

$$\begin{aligned}({}^{AB}{}_{a}\nabla^{-(\alpha,\mu)}f)(t) &= \frac{1-\alpha}{B(\alpha)}\left({}_{a}\nabla^{-(1-\mu)}f\right)(t)\\ &+\frac{\alpha}{B(\alpha)}\left({}_{a}\nabla^{-(1-\mu+\alpha)}f\right)(t), \quad t \in \mathbb{N}_a.\end{aligned} \tag{22}$$

2. Let $f:{}_{b}\mathbb{N} \to \mathbb{R}, \alpha \in (0,\frac{1}{2}), \lambda = \frac{-\alpha}{1-\alpha}$, and let B be a normalization function such that $B(0) = B(1) = 1$. Then the generalized AB nabla right fractional sum is defined as follows:

$$\begin{aligned}({}^{AB}\nabla_{b}^{-(\alpha,\mu)}f)(t) &= \frac{1-\alpha}{B(\alpha)}\left(\nabla_{b}^{-(1-\mu)}f\right)(t)\\ &+\frac{\alpha}{B(\alpha)}\left(\nabla_{b}^{-(1-\mu+\alpha)}f\right)(t), \quad t \in_{b} \mathbb{N}.\end{aligned} \tag{23}$$

The operators ${}_{a}\nabla^{-\alpha}$ and $\nabla_{b}^{-\alpha}$ are defined as in (6) and (7), respectively.

3. Summation by Parts and Euler-Lagrange Equations with Generalized M–L Function of One Parameter

The study of discrete fractional calculus has gained increasing attention over the last decades, motivated by its applications in variational principles and dynamical systems. A thorough understanding of the subject requires not only the formal definitions of fractional sums and differences but also an awareness of the historical development of the variational principles in the discrete fractional setting. This section reviews some integration by parts formulas within the scope of discrete fractional calculus, followed by the formalism of discrete fractional Euler–Lagrange equations with ABC and ABR nabla fractional differences. Finally, the review identifies how the thesis extends previous findings to derive different types of Euler–Lagrange equations.

3.1 Discrete Fractional Summation by Parts Formulas For ABC Fractional Differences

Integration by parts formula plays an important role in classical continuous calculus, and it is no exception in continuous fractional calculus. In continuous fractional calculus there are various integration by parts formulas due to having several definitions of the fractional-order integral. The same is applied in discrete fractional calculus, as the definitions of the fractional sums and differences are not unique. Thus, different fractional summation by parts expressions have been obtained. Notice that the integration becomes summation in the discrete setting. These summation by parts formulas are essential in discrete fractional calculus of variations, since they allow one to move the fractional operator from one function to another and hence free the function that appears due to doing perturbations around the extremum of a functional. Two nabla fractional summation by parts formulas are reviewed in this subsection.

First, the summation by parts formula for the ABC nabla left fractional difference with M-L function of one parameter is discussed in the next theorem.

Theorem 3.1 [23] Assume that f and g are two functions whose domains are $\mathbb{N}_a^b$, with $a < b$ and $a \equiv b(mod 1)$. Assume that $\alpha \in (0,1)$, then the following summation by parts identity holds:

$$\begin{aligned}\sum_{t=a}^{b-1} f(t)({}_{a-1}^{ABC}\nabla^{\alpha} g)(t) &= \sum_{t=a}^{b-1} g(t)({}^{ABR}\nabla_{b-1}^{\alpha} f)(s) \\ &+g(\rho(s))\frac{B(\alpha)}{1-\alpha}\sum_{t=s}^{b-1} E_{\overline{\alpha}}(\lambda, t-\rho(s))f(t)|_{s=a}^{b}.\end{aligned} \tag{24}$$

Second, the summation by parts formula for the ABC nabla right fractional difference with M-L function of one parameter is presented in the following theorem.

Theorem 3.2 [23] Assume that f and g are two functions whose domains are $\mathbb{N}_a^b$, with $a < b$ and $a \equiv b(mod 1)$. Suppose that $\alpha \in (0,1)$, then the following summation by parts formula applies:

$$\begin{aligned}\sum_{t=a+1}^{b} f(t)({}^{ABC}\nabla_{b+1}^{\alpha} g)(t) &= \sum_{t=a+1}^{b} g(t)({}_{a+1}^{ABR}\nabla^{\alpha} f)(t) \\ &-g(\sigma(s))\frac{B(\alpha)}{1-\alpha}\sum_{t=a+1}^{s} E_{\overline{\alpha}}(\lambda, s-\rho(t))f(t)|_{s=a}^{b}.\end{aligned} \tag{25}$$

Having reviewed the main definitions of delta and nabla fractional sums and differences, along with various summation by parts formulas, one can now consider their role in variational calculus. The following subsection presents two versions of the Euler–Lagrange equations depending on the type of fractional difference used in the functional.

3.2 Discrete Fractional Nabla Euler-Lagrange Equations with ABC and ABR Fractional Differences

The framework of calculus of variations was first extended to the continuous fractional setting by Agrawal in 2002 [31]. This approach has been further explored by various researchers such as Frederico and Torres [32]. This extension is achieved by replacing integer-order derivatives with various types of fractional derivatives.

Regarding the discrete fractional setting, Bohner [33] appears to be the first to study the variational problem in the discrete setting, replacing the usual derivative with the delta derivative. Subsequently, discrete fractional variational problems on general time scales have been investigated by other researchers [34].

It seems that the first delta fractional Euler-Lagrange equation on the domain $\mathbb{N}_a$ was first derived by Atici and Sengul in [13]. Another delta Euler-Lagrange equation, with the more recent and refined definition of the right delta fractional difference, is studied by Abdeljawad and Baleanu [16]. Turning to the nabla fractional Euler-Lagrange equation, Abdeljawad in [27] derived the nabla fractional Euler-Lagrange equations when the Lagrangian includes the nabla Riemann and Caputo left fractional differences. As new definitions of nabla fractional difference were obtained, new Euler-Lagrange equations were derived as well. Abdeljawad and Baleanu in 2016 and 2017 [23, 24], found the discrete nabla Euler-Lagrange equations for Lagrangians that depend on the ABC and CFC nabla fractional differences, respectively.

The next theorems present the nabla fractional Euler-Lagrange equation when the Lagrangian contains the ABC nabla left fractional difference as done in [23].

Theorem 3.3 [23] Assume that

$$S = \{y: \mathbb{N}_{a-1}^{b-1} \to \mathbb{R}: y(a-1) = A, y(b-1) = B\}$$

with $\alpha \in (0,1)$, $a, b \in \mathbb{R}$, $a < b$, and $a \equiv b(\mathrm{mod} 1)$. Moreover, suppose that

$$J(y)=\sum_{t=a}^{b-1}L(t,y^{\rho}(t),{}_{a-1}^{ABC}\nabla^{\alpha}y(t)),$$

where $L:\mathbb{N}_{a-1}^{b-1}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Let $y\in S$ be a local extremum of the given discrete functional. Then the discrete nabla Euler-Lagrange equation is given by:

$$L_1(t)+{}^{ABR}\nabla_{b-1}^{\alpha}L_2(t)=0,\quad \text{forall} t\in\mathbb{N}_{a-1}^{b-1},\qquad (26)$$

where $L_1(t)=\frac{\partial L}{\partial y^{\rho}}(t)$ and $L_2(t)=\frac{\partial L}{\partial({}_{a-1}^{ABC}\nabla^{\alpha}y)}(t)$.

Allowing the Lagrangian to include the ABC nabla right fractional difference results in a new nabla fractional Euler-Lagrange equation:

Theorem 3.4 [23] Consider the following functional

$$J(f)=\sum_{t=a+1}^{b}L(t,y^{\sigma}(t),{}^{ABC}\nabla_{b+1}^{\alpha}y(t)),$$

where $L:\mathbb{N}_{a+1}^{b+1}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Suppose that

$$S=\{y:\mathbb{N}_{a+1}^{b+1}\to\mathbb{R}:y(a+1)=A,y(b+1)=B\}$$

with $\alpha\in(0,1)$, $a,b\in\mathbb{R}$, $a<b$, and $a\equiv b(\text{mod}1)$. Let $y\in S$ be a local optimizer of J, then the Euler-Lagrange condition takes the form:

$$L_1(t)+{}_{a+1}^{ABR}\nabla^{\alpha}L_2(t)=0,\quad \text{forall} t\in\mathbb{N}_{a+1}^{b},\qquad (27)$$

where $L_1(t)=\frac{\partial L}{\partial y^{\sigma}}(t)$ and $L_2(t)=\frac{\partial L}{\partial({}^{ABC}\nabla_{b+1}^{\alpha}y)}(t)$.

.

In Theorems 3.3 and 3.4, the ABC nabla fractional differences used are associated with the one-parameter M–L function defined in [23]. All these results can be generalized by using the generalized ABC fractional difference with the generalized three-parameter M–L function defined in [25]. In order to derive the Euler–Lagrange equations with the generalized ABC and ABR nabla fractional differences, one needs summation by parts formulas. That is what will be done in the next section.

4. Summation by Parts and Euler-Lagrange Equations with Generalized M–L Function of Three Parameters Results

Some summation-by-parts formulas are given in the first subsection; these formulas are essential for establishing the main results of this thesis. Second, the generalized Euler–Lagrange equations obtained through the use of the generalized ABC nabla left and right fractional differences, in which the kernel is the three-parameter generalized M–L function, instead of the ABC nabla fractional difference with the one-parameter nabla M–L function studied in [23].

4.1 Summation by Parts for Discrete Generalized ABC Nabla Fractional Differences

This subsection presents our first results, which are the summation-by-parts formulas corresponding to the generalized ABC nabla left and right fractional differences with M-L function of three parameters, which can be seen as generalizations of the formulas obtained by Abdeljawad and Baleanu in [23], where they used the discrete nabla M–L function of one parameter.

Theorem 4.1 (Summation by Parts Formula for the Generalized ABC Nabla Left Fractional Difference) For the functions $f, g: \mathbb{N}_a^b \to \mathbb{R}$, with $a \equiv b(mod 1)$ and $0 < \alpha < \frac{1}{2}$, one has the following formula:

$$\sum_{s=a+1}^{b-1} g(s)\left({}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} f\right)(s) = \sum_{s=a+1}^{b-1} f(s)\left({}^{ABR}\nabla_b^{\alpha,\mu,\gamma} g\right)(s) + f(s)\left(\sum_{r=s+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, g(r)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-s)\right)\Big|_{s=a}^{b-1}. \tag{28}$$

Proof. By the definition of the generalized ABC nabla left fractional sum (20),

$$\text{L.H.S.} = \sum_{s=a+1}^{b-1} g(s)\, \frac{B(\alpha)}{1-\alpha} \sum_{r=a+1}^{s} E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, s-\rho(r))\nabla_r f(r).$$

Since $a+1 \le s \le b-1$ and $a+1 \le r \le s$,

Then $a+1 \le r \le s \le b-1$.

Hence, $a+1 \le r \le b-1$ and $r \le s \le b-1$.

changing the order of summation yields

$$\text{L.H.S.} = \sum_{r=a+1}^{b-1} \nabla_r f(r)\left(\sum_{s=\rho(r)+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, s-\rho(r))\right).$$

By the summation by parts formula (2) of nabla difference,

$$\text{L.H.S.} = f(r)\left(\sum_{s=r+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, s-r)\right)\Big|_{r=a}^{b-1} - \sum_{r=a+1}^{b-1} f(r)\, \nabla_r\left(\sum_{s=r+1}^{b-1} \frac{B(\alpha)}{1-\alpha} g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, s-r)\right).$$

Since $\nabla f(r) = \Delta f(r-1)$

$$\mathrm{L.H.S.} = \sum_{r=a+1}^{b-1} f(r)\left(\Delta_r \frac{-B(\alpha)}{1-\alpha} \sum_{s=r-1+1}^{b-1} g(s)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, s-\rho(r))\right)$$
$$+ f(r)\left(\sum_{s=r+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, g(s)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, s-r)\right)|_{r=a}^{b-1}.$$

And using the definition of the generalized ABR nabla right fractional difference (19), it follows:

$$\mathrm{L.H.S.} = \sum_{r=a+1}^{b-1} f(r)\left({}^{ABR}\nabla_b^{\alpha,\mu,\gamma} g\right)(r) + f(r)\left(\sum_{s=r+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, g(s)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, s-r)\right)|_{r=a}^{b-1} = \mathrm{R.H.S.}$$

Note, the right-hand side is reached by changing the name of the variables.

Now, the summation-by-parts formula for the generalized ABC nabla right fractional difference is stated in the following theorem:

Theorem 4.2 (Summation-by-Parts Formula for the Generalized ABC Nabla Right Fractional Difference) Consider the real-valued be functions f and g whose domains are $\mathbb{N}_a^b$, on which $a \equiv b(mod 1)$. Suppose that $\alpha \in (0, \frac{1}{2})$, the formula below is then valid:

$$\sum_{s=a+1}^{b-1} g(s)\left({}^{ABC}\nabla_b^{\alpha,\mu,\gamma} f\right)(s)$$
$$= \sum_{s=a+1}^{b-1} f(s)\left({}^{ABR}{}_a\nabla^{\alpha,\mu,\gamma} g\right)(s)\left(\sum_{r=a+1}^{s-1} \frac{-B(\alpha)}{1-\alpha}\, g(r)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, s-r)\right)|_{s=a+1}^{b}. \tag{29}$$

Proof. Starting with the definition of the generalized ABC nabla right fractional sum (21),

$$\mathrm{L.H.S.} = \sum_{s=a+1}^{b-1} g(s)\, \frac{-B(\alpha)}{1-\alpha} \sum_{r=s}^{b-1} E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, r-\rho(s))\Delta_r f(r).$$

Since $a+1 \le s \le b-1$ and $s \le r \le b-1$,

Then $a+1 \le s \le r \le b-1$.

Hence, $a+1 \le r \le b-1$ and $a+1 \le s \le r$.

Thus, by changing the order of summation, one obtains

$$\mathrm{L.H.S.} = \sum_{r=a+1}^{b-1} \Delta_r f(r)\left(\sum_{s=a+1}^{r} \frac{-B(\alpha)}{1-\alpha}\, g(s)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, r-\rho(s))\right)$$

$$= \sum_{r=a+1}^{b-1} \Delta_r f(r)\left(\sum_{s=a+1}^{\sigma(r)-1} \frac{-B(\alpha)}{1-\alpha}\, g(s)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, \sigma(r)-s)\right).$$

By the summation by parts formula (1) of delta difference,

$$\mathrm{L.H.S.} = f(r)\left(\sum_{s=a+1}^{r-1} \frac{-B(\alpha)}{1-\alpha}\, g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-s)\right)\Big|_{r=a+1}^{b} + \sum_{r=a+1}^{b-1} f(r)\, \Delta_r\left(\sum_{s=a+1}^{r-1} \frac{B(\alpha)}{1-\alpha} g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-s)\right).$$

Since $\Delta f(r) = \nabla f(r+1)$

$$\begin{aligned}\mathrm{L.H.S.} = \sum_{r=a+1}^{b-1} f(r)\left(\nabla_r \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{r} g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-\rho(s))\right)\\ + f(r)\left(\sum_{s=a+1}^{r-1} \frac{-B(\alpha)}{1-\alpha}\, g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-s)\right)\Big|_{r=a+1}^{b}.\end{aligned}$$

Using the definition of the generalized ABR nabla left fractional difference (18), one obtains

$$\mathrm{L.H.S.} = \sum_{r=a+1}^{b-1} f(r)\,\left({}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma} g\right)(r) + f(r)\left(\sum_{s=a+1}^{r-1} \frac{-B(\alpha)}{1-\alpha}\, g(s)\, E^{\gamma}_{\overline{\alpha,\mu}}(\lambda, r-s)\right)\Big|_{r=a+1}^{b} = \mathrm{R.H.S.}$$

Note, the right-hand side is reached by changing the name of the variables.

Since the required summation-by-parts formulas have been presented, the discrete nabla fractional difference Euler-Lagrange equations with the generalized M-L function of three parameters will be studied in the next subsection.

4.2 Discrete Fractional Nabla Euler-Lagrange Equations with Generalized ABC and ABR Fractional Differences

the Euler-Lagrange equation for the case in which the Lagrangian contains the generalized ABC nabla left fractional difference is studied in the next Theorem.

Theorem 4.3 [Euler–Lagrange Equation with Generalized ABC Nabla Left Fractional Difference in the Lagrangian] Let $S = \{y\colon \mathbb{N}_a^b \to \mathbb{R},\ \ y(a) = A, y(b-1) = B\}$, where a, b are real numbers, such that $a \equiv b(mod 1)$ and $a < b$. Assume that $\alpha \in (0, \frac{1}{2})$ Consider the following discrete fractional functional

$$J(y) = \sum_{t=a+1}^{b-1} L(t, y(t),\ {}^{ABC}{}_{a}\nabla^{\alpha,\mu,\gamma} y(t)),$$

where $L\colon \mathbb{N}_a^b \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Suppose that $y \in S$ is a local extremum of J. Then the y fulfills the following Euler-Lagrange condition

$$L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t) = 0, \quad \forall t \in \mathbb{N}_{a+1}^{b-1}, \tag{28}$$

where $L_1(t) = \frac{\partial L(t)}{\partial y}$ and $L_2(t) = \frac{\partial L(t)}{\partial ({}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} y)}$.

Proof. Let $(X, ||.||)$ be the space of functions, on which the discrete functional J is defined. Without loss of generality, suppose that $y \in S$ is a local maximum of the discrete functional J. Therefore, there exists $\varepsilon > 0$ for which

$$J(\hat{y}) - J(y) \leq 0, \qquad (29)$$

for all $\hat{y} \in S$, with $\| \hat{y} - y \| < \varepsilon$.

For any $\hat{y} \in S,$ thereisan$\eta \in H := \{y : \mathbb{N}_a^b \to \mathbb{R}, y(a) = y(b-1) = 0\}$ such that $\hat{y} = y + \epsilon\eta$.

Since that the ABC nabla fractional difference is linear then

$${}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\hat{y}(t) = {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} y(t) + \varepsilon\, {}^{\alpha,\mu,\gamma}{}_a\nabla^{\alpha}\eta(t).$$

Also,

$$L\left(t, \hat{y}(t), {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\hat{y}(t)\right) = L\left(t, y(t) + \varepsilon\eta(t), {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} y(t) + \varepsilon\, {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\eta(t)\right).$$

By the ε–Taylor's theorem,

$$\begin{aligned} & L\left(t, \hat{y}(t), {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\hat{y}(t)\right) \\ & \quad = L\left(t, y(t), {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} y(t)\right) + \varepsilon\left[\eta(t)\frac{\partial L}{\partial y}(t) + \left({}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\eta(t)\right)\frac{\partial L}{\partial\left({}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma} y\right)}(t)\right] \\ & \quad + O(\varepsilon^2). \end{aligned}$$

Therefore,

$$J(\hat{y}) - J(y) = \varepsilon \sum_{t=a+1}^{b-1} \left[\eta(t)L_1(t) + \left({}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\eta(t)\right)L_2(t)\right] + O(\varepsilon^2).$$

The sign of $J(\hat{y}) - J(y)$ is determined by the sign of the first variation $\delta J(\eta, y)$. But, if $\eta \in H$ then $-\eta \in H$ as well. And, $\delta J(\eta, y) = -\delta J(-\eta, y)$, which implies that the first variation should be zero for all $\eta \in H$. That is to say:

$$\delta J(\eta, y) = \sum_{t=a+1}^{b-1} \left[\eta(t)\, L_1(t) + {}^{ABC}{}_a\nabla^{\alpha,\mu,\gamma}\eta(t)\, L_2(t)\right] = 0, \ \forall \eta \in H. \qquad (30)$$

By applying the summation by parts formula (28), it follows that for all η in H:

$$\delta J(\eta, y) = \sum_{t=a+1}^{b-1} \eta(t)\left[L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t)\right] + \eta(t)\left(\sum_{r=t+1}^{b-1} \frac{B(\alpha)}{1-\alpha}\, L_2(r)\, E_{\overline{\alpha,\mu}}^{\gamma}(\lambda, r-t)\right)\Big|_{t=a}^{b-1} = 0.$$

Applying the boundary conditions $\eta(a) = \eta(b-1) = 0$ leads to

$$\delta J(\eta, y) = \sum_{t=a+1}^{b-1} \eta(t)\left[L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t)\right] = 0 \quad \forall \eta \in H$$

Consider a special choice of η:

$$\eta(t) = \begin{cases} 1, & t = a+1 \\ 0, & otherwise \end{cases}$$

This implies that

$$L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t) = 0, \quad \text{at } t = a+1.$$

Another choice of η is:

$$\eta(t) = \begin{cases} 1, & t = a+2 \\ 0, & otherwise \end{cases}$$

This implies that

$$L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t) = 0, \quad \text{at } t = a+2.$$

Go on on the same manner until the last choice of η:

$$\eta(t) = \begin{cases} 1, & t = b-1 \\ 0, & otherwise \end{cases}$$

This implies that

$$L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t) = 0, \quad \text{at } t = b-1.$$

In other words: one deduces that

$$L_1(t) + {}^{ABR}\nabla_b^{\alpha,\mu,\gamma} L_2(t) = 0, \quad \forall\, t \in \mathbb{N}_{a+1}^{b-1}.$$

The following theorem is devoted to the case in which the Lagrangian is a function of the generalized ABC nabla right fractional difference. The previous theorem together with the following theorem can be viewed as a generalization of the work of Abdeljawad and Baleanu in [23].

Theorem 4.4 [Euler–Lagrange Equation with Generalized ABC Nabla right Fractional Difference in the Lagrangian] Let J take the following form:

$$J(y) = \sum_{t=a+1}^{b-1} L\left(t, y(t), {}^{ABC}\nabla_b^{\alpha,\mu,\gamma} y(t)\right),$$

where $L: \mathbb{N}_a^b \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a, b are real values such that $a < b$ and $a \equiv b(\text{mod}1)$. Assume further that $0 < \alpha < \frac{1}{2}$ and

$$S = \{y: \mathbb{N}_a^b \to \mathbb{R}, \quad y(a+1) = A, y(b) = B\}$$

If $y \in S$ is a local extremum of J, then y is subject to the following Euler-Lagrange condition:

$$L_1(t)+{}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}L_2(t)=0,\quad \forall t\in\mathbb{N}_{a+1}^{b-1},\qquad (31)$$

where $L_1(t)=\frac{\partial L(t)}{\partial y}$ and $L_2(t)=\frac{\partial L(t)}{\partial({}^{ABC}{}_{a}\nabla^{\alpha,\mu,\gamma}y)}$.

Proof. Assume that J is a discrete fractional functional with domain $(X,\ ||.||)$. Assume that $y\in S$ is a local maximum of the functional J; the case of a minimum follows analogously. Then, for some $\varepsilon>0$

$$J(\hat{y})-J(y)\le 0,\qquad (32)$$

for all $\hat{y}\in S$, with $\parallel\hat{y}-y\parallel<\varepsilon$.

For any $\hat{y}\in S$, thereisan$\eta\in H:=\{y:\mathbb{N}_a^b\to\mathbb{R}, y(a+1)=y(b)=0\}$ for which $\hat{y}=y+\epsilon\eta$.

As in Theorem 4.3, the first variation is found by applying ε–Taylor's theorem. Then, to ensure that (34) is satisfied, one obtains the following:

$$\delta J(\eta,y)=\sum_{t=a+1}^{b-1}\left[\eta(t)\,L_1(t)+{}^{ABC}\nabla_{b}^{\alpha,\mu,\gamma}\eta(t)\,L_2(t)\right]=0,\quad\forall\eta\in H.\qquad (33)$$

Summation by parts formula (29) leads to the following:

$$\delta J(\eta,y)=\sum_{t=a+1}^{b-1}\eta(t)[L_1(t)+{}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}L_2(t)]+\eta(t)\left(\sum_{r=a+1}^{t-1}\frac{-B(\alpha)}{1-\alpha}\,L_2(r)\,E_{\alpha,\mu}^{\gamma}(\lambda,t-r)\right)\Big|_{t=a+1}^{b},$$

$$\forall\eta\in H.$$

But, $\eta(a+1)=\eta(b)=0$. Thus,

$$\delta J(\eta,y)=\sum_{t=a+1}^{b-1}\eta(t)[L_1(t)+{}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}L_2(t)]=0,\quad\forall\eta\in H.$$

Since η is arbitrary, certain choices of α similar to the proof of Theorem 4.3 yield

$$L_1(t)+{}^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}L_2(t)=0,\quad\forall\, t\in\mathbb{N}_{a+1}^{b-1}.$$

Finally, to illustrate the results, two examples of physical interest, corresponding to Theorems 4.3 and 4.4, are presented.

Example 4.1 Consider the following discrete fractional functional

$$J(y)=\sum_{t=a+1}^{b-1}\frac{1}{2}({}^{ABC}{}_{a}\nabla^{\alpha,\mu,\gamma}y(t))^2-V(y,t).$$

Let $\alpha\in(0,\frac{1}{2}), y(a)=A,$ and$y(b-1)=B,$ where$A,B\in\mathbb{R}$ and $m\in\mathbb{R}$ is any constant. By Theorem 4.3, the Euler-Lagrange equation for this functional is given by

$${}^{ABR}\nabla_{b}^{\alpha,\mu,\gamma}({}^{ABC}{}_{a}\nabla^{\alpha,\mu,\gamma}y)(t)-\frac{\partial V}{\partial y}(t)=0,\forall t\in\mathbb{N}_{a+1}^{b-1}.$$

Example 4.2 Let

$$J(y)=\sum_{t=a+1}^{b-1}\frac{1}{2}\left({}^{ABC}\nabla_{b}^{\alpha,\mu,\gamma}y(t)\right)^{2}-V(y,t),$$

with $0<\alpha<\frac{1}{2}, y(a+1)=A, \text{and} y(b)=B$, where A, B are real constants, and $m\in\mathbb{R}$ is any constant. By Theor m 4.4, the corresponding Euler-Lagrange equation reads

$$^{ABR}{}_{a}\nabla^{\alpha,\mu,\gamma}\left({}^{ABC}\nabla_{b}^{\alpha,\mu,\gamma}y\right)(t)-\frac{\partial V}{\partial y}(t)=0,\forall t\in\mathbb{N}_{a+1}^{b-1}.$$

5. Conclusion and Future Directions

This paper is devoted to the study of discrete fractional variational problems involving ABC and ABR nabla fractional differences with the generalized M–L kernels. It is known that integration by parts is essential in calculus of variations as well as in fractional calculus of variations. The same is applied in discrete fractional calculus of variations thus new summation by parts formula for ABC nabla left and right fractional differences are obtained to use them in getting new discrete nabla Euler–Lagrange equations.

New discrete fractional Euler–Lagrange equations were derived for functionals involving the generalized ABC nabla left and right fractional differences with the generalized M–L function of three parameters. These results extend already existing Euler–Lagrange equations with ABC and ABR fractional differences and provide a more general formulation within a broader setting. Some illustrative examples of physical interest were also established to demonstrate the viability of the obtained theoretical results.

Even though the present work establishes several theoretical results, a number of important research directions is still open for further investigation. One natural extension is to study discrete fractional variational problems with delay terms since such terms frequently arise in applications, and can significantly affect the stability and dynamical behavior of solutions. The problems studied in this thesis are formulated with fixed end-point boundary conditions thus future work may address discrete fractional variational problems with more general types of constraints, including isoperimetric constraints by applying Lagrange multiplier techniques. Further promising extensions may include the development of ABC and ABR delta fractional sums and differences, together with the derivation of the corresponding Euler–Lagrange equations.

ASA, sarcasm detection, and real-time deployment on edge devices.

8. Declarations

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Ethical consideration

Not applicable.

Consent to participate

Not applicable

Conflicts of interest

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